

# ELECTROMAGNETIC FIELDS IN KERR-SCHILD SPACE-TIMES

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Making use of twistor structures and the Kerr theorem for shear-free null geodesic congruences, an infinite family of electromagnetic fields satisfying the homogeneous Maxwell equations in flat Minkowski and the associated curved Kerr-Schild backgrounds is obtained for any such congruence in a purely algebraic way. Simple examples of invariant axisymmetric Maxwell fields are presented.

## 1. Introduction

Rather unexpectedly, one observes a growing interest in solutions of the most investigated, seemingly simple and important equations of classical field theory, the Maxwell linear homogeneous equations in vacuum. It occurs that these possess various kinds of solutions with different topology of field lines [1, 2], with extended singularities of complicated shape and temporal dynamics [3, 4, 5], multi-valued complexified solutions etc. The simplest example of the latter is provided by the electromagnetic field of the Kerr-Newman solution in general relativity (which is quite the same in the curved Kerr-Newman background and in flat Minkowski space).

To obtain complicated solutions to the Maxwell equations, various algebraic methods can be used, which involve twistor structures or generating procedures starting from a solution to some (linear or nonlinear) “master equation” (the concepts of a superpotential or “hidden nonlinearity”, respectively). In the later case, one obtains natural restrictions, a sort of “selection rules”, on characteristics of the associated Maxwell fields, in particular, on admissible values of the electric charge of isolated singularities [1, 5, 6].

It is especially noteworthy that, in many cases, solutions to the Maxwell equations in flat space-time  $\eta_{\mu\nu}$  are exactly invariant under a deformation of the metric to the following *Kerr-Schild form*:

$$g_{\mu\nu} = \eta_{\mu\nu} + H(x)k_\mu k_\nu, \quad (1)$$

$k_\mu(x)$  being a null 4-vector field. Indeed, for a determinant of any metric of the form (1) one obtains  $\sqrt{-g} \equiv 1$ , whereas the condition

$$g^{\mu\rho}g^{\nu\lambda}F_{\rho\lambda} = \eta^{\mu\rho}\eta^{\nu\lambda}F_{\rho\lambda} := F^{\mu\nu} \quad (2)$$

is fulfilled iff the 4-vector  $k_\mu$  is an eigenvector of the field strength tensor  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , that is, iff

$$F_{\mu\nu}k^\nu \sim k_\mu. \quad (3)$$

The well known examples of such an invariance of Maxwell field under a Kerr-Schild deformation of the flat metric are the ordinary Coulomb field in the Reissner-Nordström background or the Kerr-Newman field mentioned above.

In Section 2, we describe a procedure which makes use of twistor structures and allows us to associate a whole family of electromagnetic fields *with any shear-free null congruence (SFNC)* of rays in Minkowski space. We demonstrate that the condition (3) *holds identically for all such fields* if  $k_\mu$  is a null 4-vector field tangent to the SFNC under consideration. Consequently, any SFNC gives rise to a family of Kerr-Schild metrics (with different  $H(x)$  in (1)) and a family of electromagnetic fields identically satisfying Maxwell equations

$$\partial_\nu(\sqrt{-g}F^{\mu\nu}) = 0 \quad (4)$$

in the corresponding curved background. In addition, it is well known that the scalar “gravitational potential”  $H(x)$  may be often fitted to ensure that the Kerr-Schild metric satisfies the vacuum or electrovacuum Einstein equations.

In Section 3, we present simple examples of invariant electromagnetic fields associated with SFNC and inheriting the spherical or axial symmetry of the congruence. In all such cases the newly found fields correspond to the previously determined ones (which are known to possess the invariance property). In general, however, the invariant fields introduced in the paper are new and can open a way to complicated electrovacuum solutions with metrics of the Kerr-Schild type.

## 2. Shear-free null congruences and the associated Maxwell fields in flat and curved background

According to the *Kerr theorem* [7], any SFNC in Minkowski space may be obtained from the generating equation

$$\Pi(G, \tau^1, \tau^2) \equiv \Pi(G, wG + u, vG + \bar{w}) = 0, \quad (5)$$

where  $\Pi$  is an analytical function of three complex arguments  $\{G, \tau^1, \tau^2\}$  representing the components of a *projective twistor*. The latter is associated with the coordinates  $u, v = t \pm z$ ,  $w, \bar{w} = x \mp iy$  of points in Minkowski space through the Penrose's *incidence relation*

$$\tau^1 = wG + u, \quad \tau^2 = vG + \bar{w}. \quad (6)$$

Resolving (5) with respect to the unknown  $G$ , one comes to a multi-valued field  $G = G(u, v, w, \bar{w})$ . It is easy to demonstrate (via differentiation of (5)) that any continuous branch of this multi-valued field satisfies the determining equations of a SFNC,

$$\partial_w G = G \partial_u G, \quad \partial_v G = G \partial_{\bar{w}} G, \quad (7)$$

and therefore represents the principal spinor of the latter. Thus, one has a one-to-one correspondence between the SFNCs and the twistor functions  $\Pi(G, \tau^1, \tau^2)$  (precisely, the surfaces  $\Pi = 0$  in the  $\mathbb{CP}^3$  projective twistor space).

From (7) the complex eikonal equation follows immediately,

$$|\nabla G|^2 := \partial_u G \partial_v G - \partial_w G \partial_{\bar{w}} G = 0, \quad (8)$$

and, as the integrability condition of (7), one obtains then the d'Alembert equation [8]

$$\square G := \partial_u \partial_v G - \partial_w \partial_{\bar{w}} G = 0. \quad (9)$$

One might thus suspect that any SFNC gives rise to a (complexified) Maxwell field, with  $G$  being a sort of super-potential. Indeed, in [3, 6, 8] such a field has been obtained, with its strength expressed through the second-order derivatives of  $G$ . Electromagnetic fields of this type possess a number of the afore-mentioned peculiar properties. However, these fields do not preserve their form under the Kerr-Schild deformation of metric. To ensure such an invariance, we propose below another expression for Maxwell fields to be associated with arbitrary SFNC.

Specifically, consider the following simple ansatz for the components of the spintensor  $\varphi_{A'B'}$ ,  $A', B' = 1', 2'$  of electromagnetic field strength:

$$\varphi_{1'1'} = \partial_u \partial_u F, \quad \varphi_{1'2'} = \varphi_{2'1'} = \partial_w \partial_u F, \quad \varphi_{2'2'} = \partial_w \partial_w F, \quad (10)$$

where  $F = F(G)$  is an arbitrary (holomorphic) function of the principal spinor component  $G(X)$ .

Taking into account that  $X = \{X^{AA'}\} = \{u, w, \bar{w}, v\}$ , it is easy to check that the field (10) in fact satisfies the homogeneous Maxwell equations

$$\partial^{AA'} \varphi_{A'B'} = 0 \quad (11)$$

provided the eikonal (8) and d'Alembert (9) equations hold both together for the function  $G$ .

Now, let us prove that all fields (10) associated with a SFNC always obey the eigenvector condition (3). To do so, let us rewrite the latter in the equivalent 2-spinor form

$$\varphi_{A'B'} \xi^{A'} \xi^{B'} = 0, \quad (12)$$

where  $\xi_{A'}$  is the principal spinor of the SFNC in the gauge  $\xi_{1'} = 1$ ,  $\xi_{2'} = G$ . Substituting the components

(10) into the above condition (12) and transforming the l.h.s. of the latter, one obtains

$$G \partial_u (F' M) - \partial_w (F' M) - (F' M) \partial_u G = 0, \quad (13)$$

where  $F' := dF(G)/dG$ ,  $M := G \partial_u G - \partial_w G$ . However,  $M \equiv 0$  on account of the first of the SFNC determining equations (7), so that the eigenvector condition (12) is identically satisfied for the fields (10). This means that *there exists an infinite family of electromagnetic fields (10) associated with a SFNC of a general form, which all satisfy the homogeneous Maxwell equations in Minkowski space and in the corresponding curved Kerr-Schild space.*

Some remarks are in order here.

1. In a number of papers E.T. Newman et al. and A.Ya. Burinskii (see, e.g., [9, 10]) had been working with electromagnetic fields generated by a pointlike charge moving in real or complexified Minkowski space. These fields indeed satisfy the invariance condition (3). However, even for this particular case, to obtain such fields, explicit integration of the Maxwell equations was required.

2. To simplify the exposition, nearly all the above constructions have been presented in a particular gauge. One understands, nonetheless, that the formalism can be easily transformed into a manifestly Lorentz invariant form.

3. For the newly found fields, the theorem on “quantization of the electric charge” for isolated singularities [5] remains valid and will be reproduced elsewhere (see also the examples below).

4. Remarkably, symmetries of the electromagnetic fields (10) for a general form of  $F(G)$  can be weaker than those of the SFNC which they are generated from. For example, the Kerr axisymmetric congruence can give rise to electromagnetic fields devoid of any symmetries at all. To preserve the axial symmetry of a SFNC, a particular form of the function  $F(G)$ , namely  $F(G) = G^{-1}$ , must be used (see below).

### 3. Some examples of invariant Maxwell fields associated with a SFNC

1. Consider first a static, spherically symmetric SFNC and the associated fields corresponding to the following form of generating equation (5):

$$\Pi = G \tau^1 - \tau^2 \equiv w G^2 - 2zG + \bar{w} = 0, \quad z = (u-v)/2. \quad (14)$$

Resolving (14), one obtains

$$G = \frac{\bar{w}}{z \pm r}, \quad r := \sqrt{w\bar{w} + z^2} \equiv \sqrt{x^2 + y^2 + z^2}, \quad (15)$$

i.e. *the stereographic projection*  $S^2 \mapsto \mathbb{C}$  from the South or North poles, respectively. Corresponding SFNC is the radial, spherically symmetric congruence with a point singularity.

One can easily verify that, from the whole family of associated electromagnetic fields (10), only the choice  $F(G) = G^{-1}$  corresponds to a field inheriting the spherical symmetry of the congruence. Specifically, calculating the components (10) of the spintensor  $\varphi_{A'B'}$  for  $F(G) = G^{-1}$ , one gets

$$\varphi_{1'1'} = q \frac{w}{r^3}, \quad \varphi_{1'2'} = -q \frac{z}{r^3}, \quad \varphi_{2'2'} = -q \frac{\bar{w}}{r^3}, \quad (16)$$

with the constant  $q = \mp 1/4$ . The expression (16) describes the pure Coulomb field with an electric charge necessarily fixed in value (in fact this is a minimum “elementary” charge, see [5, 6] for details). Of course, the Coulomb field is invariant under the Kerr-Schild type of deformation of flat metric, i.e., under transition to the Reissner-Nordström space-time.

2. The *Kerr congruence* with twist and a ring-like singularity of radius  $a$  is known to arise from the radial one through a complex shift, say,  $z \mapsto z + Ia$ . Since all the above-presented constructions deal with complex holomorphic structures, the associated Maxwell field also follows from (16) through such a shift and coincides with the electromagnetic field of the Kerr-Newman electrovacuum solution. It is important, nonetheless, that in the same Kerr-Newman metric background, making use of different  $F(G)$ , a lot of other fields obeying the Maxwell equations could be introduced.

3. The Kerr congruence can be naturally generalized to a *nonstationary* form described in [11]. To do that, one transforms the principal spinor of the radial SFNC (15) through a *complex boost* and obtains the following form of it:

$$G = (1 + Iu) \frac{x + Iy}{(z - z_t) \pm \hat{r}}, \quad z_t := -Ia + Iut, \quad (17)$$

where  $Iu \in \mathbb{C}$  is the “imaginary velocity” parameter of a “complex boost”, and “complex distance” from the point singularity to an observation point is

$$\hat{r} = \sqrt{(z - z_t)^2 + w\bar{w}(1 + u^2)}. \quad (18)$$

The singular locus (caustic) of the SFNC determined by (17) coincides with the *branching points*  $\hat{r} = 0$  of the principal spinor and represents a *Kerr-like ring* collapsing/expanding with the physical velocity  $V = u/\sqrt{1 + u^2} < 1$ . The invariant (complex-valued) electromagnetic field associated with such a congruence through the expression (10) under the assumption  $F(G) = G^{-1}$  again reproduces the previously determined one (see Eq.(40) of the paper [11]) and manifests a number of interesting properties (including its vortex-like structure and concentration of the field along the symmetry axis).

4. We thus see that, for the simplest SFNCs, the invariant electromagnetic fields (10) contain a distinguished one, inheriting the axial symmetry of the congruence and reproducing the fields associated with the

latter through the old prescription. Generally, however (in particular, for the case of a *bisingular SFNC* [3, 8]), the newly obtained (and necessarily invariant) Maxwell fields differ from the previously considered (and generally noninvariant) ones. One can hope thus that discovery of a family of fields automatically satisfying the Maxwell equations in any given Kerr-Schild background will open the way to construction of extremely complicated electrovacuum solutions corresponding to arbitrary SFNCs.

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